

Approximation of analytical functions by sequences of k -positive linear operators

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Abstract

Approximation properties of sequences of k -positive operators, i.e. linear operators acting in the space of analytical functions and preserving the cone of functions with non-negative Taylor coefficients, are studied. Some general theorems which are valid in the space of functions that are analytical in a bounded simply connected domain are proved.

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In this paper we study the problem of approximation of analytical functions in some bounded complex domain by sequences of linear operators. Some properties of classes of analytical functions and operators acting in these classes may be found in [4,12]. The problem of approximation of an analytical function in the unit disk $|z| < 1$ by sequences of linear operators was studied in [5]. A definition of so-called k -positive operators acting on analytical functions and preserving the subspace of functions with non-negative Taylor coefficients was introduced in [5] (see also [8]) to obtain Korovkin-type approximation theorems.

Some results on the approximation of analytical functions by linear k -positive operators were also established in [1–3,6,9–11]. The problem of statistical convergence of a sequence of linear k -positive operators was studied in [3], where the author used the method of [7] and the

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corresponding estimates from [10] for operators acting in the spaces of sequences of complex numbers.

In this paper, we prove more general theorems for linear operators acting on the space of analytical functions in a simply connected bounded domain. The results we obtained are new even for the space of analytical functions in the unit disk.

Let $A(D)$ be the space of analytical functions in a bounded simply connected domain D , in which convergence is the uniform convergence in any closed part of D . Let $\phi(z)$ be any function mapping D conformally and one-to-one onto the unit disk. It is known that the system of functions $\phi^k(z)$, $k = 0, 1, 2, \dots$ is a basis in the space $A(D)$ (see [4]). That is, each function $f \in A(D)$ has the following Taylor expansion:

$$f(z) = \sum_{k=0}^{\infty} f_k \phi^k(z), \quad (1)$$

where

$$f_k = \frac{1}{2\pi i} \int_C f(z) \phi^{-k-1}(z) \phi'(z) dz \quad (2)$$

and C is any contour lying in the interior of D .

By the definition of convergence in this space, $A(D)$ becomes a Fréchet-type space, and an equivalent seminorm may be defined. For $r < 1$ we will consider the following seminorm:

$$\|f\|_{A(D),r} = \max_{|\phi(z)| \leq r < 1} |f(z)|. \quad (3)$$

The following definition is an analogue of the corresponding definition given in [5].

Definition 1. A linear operator $T : A(D) \rightarrow A(D)$ is called k -positive if T preserves the cone of functions with non-negative Taylor coefficients.

Let $f \in A(D)$ and T_n be a sequence of linear operators from $A(D)$ to $A(D)$. Then by (1) we can write the following expansion for $T_n f(z)$:

$$T_n f(z) = \sum_{k=0}^{\infty} \phi^k(z) \sum_{m=0}^{\infty} f_m T_{k,m}^{(n)}.$$

It is easy to see that the linear operator T_n is k -positive if and only if $T_{k,m}^{(n)} \geq 0$ for all k, n, m . Indeed, if $T_{k,m}^{(n)} < 0$ for some $m = m_0$, we can take a function $f^*(z) = \sum_{m=0}^{\infty} f_m z^m$ with $f_{m_0} > 0$ and $f_m = 0$ if $m \neq m_0$, which creates a contradiction with the above definition.

The following theorem on convergence of the sequence $f_n(z)$ to zero in $A(D)$ is known (see [4]).

Theorem A. In order that the sequence $f_n(z)$ tends towards zero in $A(D)$ it is necessary and sufficient that

$$f_n(z) = \sum_{k=0}^{\infty} f_{n,k} \phi^k(z), \quad \limsup_{k \rightarrow \infty} |f_{n,k}|^{\frac{1}{k}} = 1$$

for any n and

$$|f_{n,k}| \leq \epsilon_n (1 + \delta_n)^k,$$

where $\lim_{n \rightarrow \infty} \epsilon_n = \lim_{n \rightarrow \infty} \delta_n = 0$.

In order to reveal further characterizations of convergence in $A(D)$, let us introduce another auxiliary result which will be used as a basic tool in our theorems on convergence.

Theorem B. *In order that the sequence $f_n(z)$ tends towards zero in $A(D)$ it is necessary and sufficient that*

$$f_n(z) = \sum_{k=0}^{\infty} f_{n,k} \phi^k(z), \quad \limsup_{k \rightarrow \infty} |f_{n,k}|^{\frac{1}{k}} = 1,$$

and that there exists a sequence $\{t_{n,k}\}$ of positive numbers such that

$$|f_{n,k}| \leq t_{n,k} \tag{4}$$

and

(a) $t_{n,k}$ tends to zero as $n \rightarrow \infty$ for any fixed $k = 0, 1, 2, \dots$,

(b) $\limsup_{k \rightarrow \infty} t_{n,k}^{\frac{1}{k}} = 1$ for any fixed $n = 1, 2, \dots$

Proof. If (4) holds and the sequence $t_{n,k}$ satisfies (b), then

$$\sum_{k=0}^{\infty} r^k |f_{n,k}| \leq \sum_{k=0}^{\infty} r^k t_{n,k}.$$

The series $\sum_{k=0}^{\infty} r^k t_{n,k}$ converges for each $r < 1$. Therefore, for each $\varepsilon > 0$, we can find a number k_0 such that

$$\sum_{k=k_0+1}^{\infty} r^k t_{n,k} < \varepsilon;$$

moreover, condition (a) implies that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{k_0} r^k t_{n,k} = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \|f_n(z)\|_{A(D),r} = 0. \tag{5}$$

Now, let (5) holds. Then the numbers

$$\varepsilon_n = \max_{|\phi(z)| \leq r < 1} |f_n(z)|$$

tend to zero as $n \rightarrow \infty$.

Due to (2), for each $r < 1$, we have

$$\begin{aligned} |f_{n,k}| &\leq \frac{1}{2\pi} \int_{|\phi(z)|=r} |f_n(z)| |\phi(z)|^{-k-1} |\phi'(z)| |dz| \\ &\leq \varepsilon_n \int_{|u|=r} \frac{|du|}{|u|^{k+1}} \leq \frac{\varepsilon_n}{r^k}. \end{aligned}$$

Taking $t_{n,k} = |f_{n,k}|$, we obtain (a) and (b). \square

Let us denote by $A_g(D)$ the set of all analytical functions whose Taylor coefficients f_k satisfy the inequality $|f_k| \leq M_f g_k$, where M_f is a constant and g_k is an increasing sequence of

non-negative numbers such that $\limsup_{k \rightarrow \infty} g_k^{\frac{1}{k}} = 1$. Note that, for the converging series given below, g_k must also satisfy the condition $\limsup_{k \rightarrow \infty} (\sqrt{g_k} - \sqrt{g_{k-1}})^{\frac{1}{k}} = 1$. The following theorem shows that the system of three functions

$$g_\nu(z) = \sum_{k=0}^{\infty} g_k^{\frac{\nu}{2}} \phi^k(z) \quad \nu = 0, 1, 2 \quad (6)$$

is a Korovkin system in the space $A_g(D)$.

Theorem 1. Let $T_n : A(D) \rightarrow A(D)$ be a sequence of linear k -positive operators. Then

$$\lim_{n \rightarrow \infty} \|T_n f - f\|_{A(D), r} = 0 \quad (7)$$

for each function $f \in A_g(D)$ if and only if

$$\lim_{n \rightarrow \infty} \|T_n g_\nu(z) - g_\nu(z)\|_{A(D), r} = 0, \quad \nu = 0, 1, 2, \quad (8)$$

where $g_\nu(z)$ is defined as in (6).

Proof. By definition of the norm in $A(D)$, it follows that, for any function $f \in A(D)$ (see (1)), the inequality

$$\begin{aligned} \|T_n f - f\|_{A(D), r} &\leq \sum_{k=0}^{\infty} r^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} |f_m - f_k| + \sum_{k=0}^{\infty} r^k |f_k| \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right| \\ &= I'_n + I''_n \end{aligned} \quad (9)$$

holds. From condition (8) with $\nu = 0$, it is easy to see that $|\sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1|$ satisfies the conditions of Theorem B. Therefore, $\lim_{n \rightarrow \infty} I''_n = 0$. To estimate the term I'_n in (9), it is easy to verify that, for all m and k ,

$$|f_m - f_k| \leq 8M \frac{g_k^3}{\Delta_k^2(g)} (\sqrt{g_m} - \sqrt{g_k})^2, \quad (10)$$

where

$$\Delta_k(g) = \min \{ \sqrt{g_k} - \sqrt{g_{k-1}}; \sqrt{g_{k+1}} - \sqrt{g_k} \}. \quad (11)$$

On the other hand, using Theorem A and condition (8), we obtain

$$\sum_{m=0}^{\infty} (\sqrt{g_k} - \sqrt{g_{k-1}})^2 T_{k,m}^{(n)} \leq 4\varepsilon_n (1 + \delta_n)^k g_k.$$

From the last inequality and (10) we infer that

$$\begin{aligned} I'_n &= \sum_{k=0}^{\infty} r^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} |f_m - f_k| \\ &\leq 32M\varepsilon_n \sum_{m=0}^{\infty} r^k (1 + \delta_n)^k \frac{g_k^4}{\Delta_k^2(g)}. \end{aligned}$$

From the conditions on g_k , it follows that the series on the right-hand side of the last inequality converges. Therefore $\lim_{n \rightarrow \infty} I'_n = 0$, and we obtain the sufficient part of the theorem. Since the functions $g_\nu(z)$ in (6) belong to $A_g(D)$, the proof is complete. \square

Let us assume that the sequence g_k has the form

$$g_k = 1 + h_k, \quad (12)$$

where h_k is increasing. Then (6) takes the form

$$g_\nu(z) = \sum_{k=0}^{\infty} (1 + h_k)^{\frac{\nu}{2}} \phi^k(z).$$

We shall show that in this particular case the test functions $g_\nu(z)$ may be chosen as

$$\tilde{g}_\nu(z) = \sum_{k=0}^{\infty} h_k^{\frac{\nu}{2}} \phi^k(z), \quad \nu = 0, 1, 2. \quad (13)$$

Below, we assume that the following condition holds:

$$\limsup_{k \rightarrow \infty} \left(\sqrt{h_k} - \sqrt{h_{k-1}} \right)^{\frac{1}{k}} = 1. \quad (14)$$

Theorem 2. Let $T_n : A(D) \rightarrow A(D)$ be a sequence of linear k -positive operators. Then (8) is valid for each function $f \in A_g(D)$, with g_k is defined as in (12), if and only if

$$\lim_{n \rightarrow \infty} \|T_n \tilde{g}_\nu - \tilde{g}_\nu\|_{A(D),r} = 0, \quad \nu = 0, 1, 2, \quad (15)$$

where \tilde{g}_ν is defined as in (13).

Proof. For each function $f \in A_g(D)$, we can get the inequality (9). Then, as in the proof of Theorem 1, we see that $\lim_{n \rightarrow \infty} I_n'' = 0$. Therefore, it is enough simply to show that the term

$$I_n' = \sum_{k=0}^{\infty} r^k \sum_{m=0}^{\infty} T_{k,m}^{(n)} |f_m - f_k|,$$

tends to zero as $n \rightarrow \infty$.

A simple calculation gives

$$\begin{aligned} |f_m - f_k| &\leq 4Mg_k^2 \left[\left(\sqrt{h_m} - \sqrt{h_k} \right)^2 + 1 \right] \\ &\leq 4Mg_k^2 \frac{(\sqrt{h_m} - \sqrt{h_k})^2}{\Delta_k^2(h)} \left(\Delta_k^2(h) + 1 \right), \end{aligned}$$

where $\Delta_k(\cdot)$ is defined as in (11).

Since $\Delta_k^2(h) + 1 \leq g_k$, we have

$$|f_m - f_k| \leq 4Mg_k^3 \frac{(\sqrt{h_m} - \sqrt{h_k})^2}{\Delta_k^2(h)}.$$

Using Theorem B and condition (15), we have

$$\sum_{k=0}^{\infty} T_{k,m}^{(n)} \left(\sqrt{h_m} - \sqrt{h_k} \right)^2 \leq 2t_{n,k} (1 + \sqrt{h_k})^2 \leq 4t_{n,k} g_k,$$

where $t_{n,k}$ tends to zero as $n \rightarrow \infty$ for any fixed $k = 0, 1, 2, \dots$. Therefore,

$$I'_n \leq 16M \sum_{k=0}^{\infty} \frac{g_k^4}{\Delta_k^2(h)} t_{n,k} r^k.$$

By the conditions on g_k and (14) we see that $\frac{g_k^4}{\Delta_k^2(h)} t_{n,k}$ satisfies conditions (a) and (b) on $t_{n,k}$ in Theorem B. Therefore, applying Theorem B we get the desired result. \square

Theorems 1 and 2 give the Korovkin systems in the space $A_g(D)$, and all these systems consist of three functions. The following theorem shows that the sequence of functions $\phi^k(z)$, $k = 0, 1, 2, \dots$ is not a Korovkin system in $A_g(D)$. Note that for k -positive operators acting in the space $A(|z| < 1)$ this result was established in [5].

Theorem 3. For any majorant $g_k \geq 1$, $k = 0, 1, 2, \dots$, there exists a sequence of linear k -positive operators $W_n : A(D) \rightarrow A(D)$ satisfying countably many conditions,

$$\lim_{n \rightarrow \infty} \|W_n \phi^m(z) - \phi^m(z)\|_{A(D),r} = 0, \quad m = 0, 1, 2, 3, \dots, \quad (16)$$

and there exists a function $f^* \in A_g(D)$ for which

$$\limsup_{n \rightarrow \infty} \|W_n f^* - f^*\|_{A(D),r} \geq \frac{1}{1+r}. \quad (17)$$

Proof. Given the majorant g_m , we consider the sequence of operators

$$W_n f(z) = \sum_{k=0}^{\infty} \phi^k(z) \sum_{m=0}^{\infty} T_{m,k}^{(n)} f_m,$$

where

$$T_{m,k}^{(n)} = \begin{cases} 0, & \text{if } m < k \\ \delta_{k,m} + \binom{m}{k} \frac{1}{n^{k+1}} \left(\frac{n}{n+1}\right)^m g_m, & \text{if } m \geq k \end{cases}$$

and $\delta_{k,m}$ is the Kroneker symbol.

Then for each function f defined in (1) we can write

$$W_n f(z) = \sum_{k=0}^{\infty} \phi^k(z) \left\{ f_k + \frac{1}{n^{k+1}} \sum_{m=k}^{\infty} \binom{m}{k} \left(\frac{n}{n+1}\right)^m f_m g_m \right\}$$

and after a simple calculation

$$W_n f(z) = f(z) + \frac{1}{n} \sum_{k=0}^{\infty} \left(\frac{n + \phi(z)}{n+1}\right)^k f_k g_k.$$

This shows that the W_n are linear k -positive operators acting from $A(D)$ to $A(D)$. Since $\phi^m(z) = \sum_{k=0}^{\infty} \delta_{k,m} \phi^k(z)$, we have

$$W_n \phi^m(z) = \phi^m(z) + \frac{1}{n} \left(\frac{n + \phi(z)}{n+1}\right)^m g_m, \quad m = 0, 1, 2, \dots,$$

and therefore, for each $r \in (0, 1)$,

$$\max_{|\phi(z)| \leq r} |W_n \phi^m(z) - \phi^m(z)| \leq \frac{1}{n} \left(\frac{n+r}{n+1} \right)^m g_m.$$

Thus (16) holds for every $m = 0, 1, 2, \dots$

Consider the function

$$f^*(z) = \sum_{k=0}^{\infty} \frac{1}{g_k} \phi^k(z).$$

Obviously, $f^* \in A_g(D)$. At the same time,

$$\begin{aligned} W_n f^*(z) &= \sum_{k=0}^{\infty} \frac{1}{g_k} \left\{ \phi^k(z) + \frac{1}{n} \left(\frac{n+\phi(z)}{n+1} \right)^k g_k \right\} \\ &= f^*(z) + \frac{1}{n} \sum_{k=0}^{\infty} \left(\frac{n+\phi(z)}{n+1} \right)^k \\ &= f^*(z) + \frac{n+1}{n} \frac{1}{1-\phi(z)}. \end{aligned}$$

Hence (17) holds true. \square

Let b_k be an increasing sequence of positive numbers, such that $\limsup_{k \rightarrow \infty} b_k^{\frac{1}{k}} = 1$. Consider the following three functions:

$$b_\nu(z) = \sum_{k=0}^{\infty} \frac{b_k^{\frac{\nu}{2}}}{1+b_k} g_k \phi^k(z), \quad \nu = 0, 1, 2, \quad (18)$$

where g_k is defined above.

It is clear that $b_\nu(z) \in A(D)$; moreover, the functions $b_\nu(z)$ have positive Taylor coefficients.

The following theorem gives a very general result on approximation in $A(D)$.

Theorem 4. Let $T_n : A(D) \rightarrow A(D)$ be a sequence of linear k -positive operators. Then

$$\lim_{n \rightarrow \infty} \|T_n f - f\|_{A(D),r} = 0 \quad (19)$$

for each function $f \in A_g(D)$ if and only if

$$\lim_{n \rightarrow \infty} \|T_n b_\nu(z) - b_\nu(z)\|_{A(D),r} = 0, \quad \nu = 0, 1, 2. \quad (20)$$

Proof. Since $b_\nu(z) \in A_g(D)$, it is enough to prove that (20) implies (19). Let $f \in A_g(D)$. Following the proof of Theorem 1, we can write the inequality (9):

$$\|T_n f - f\|_{A(D),r} \leq I'_n + I''_n.$$

Since $g_k \geq 1$, for each $k = 0, 1, 2, \dots$, we have

$$|f_m - f_k| \leq 2M g_m \frac{1+b_m}{1+b_m} g_k.$$

Here the constant M depends on the function f and is independent of m and k . Using (11), we can write

$$\Delta_k(b) = \min \left\{ \sqrt{b_k} - \sqrt{b_{k-1}}; \sqrt{b_{k+1}} - \sqrt{b_k} \right\}.$$

Obviously, for any $m \neq k$,

$$1 + b_m \leq 4 \left(\sqrt{b_m} - \sqrt{b_k} \right)^2 \frac{1 + b_k}{\Delta_k^2(b)}. \quad (21)$$

Therefore,

$$|f_m - f_k| \leq C_k \frac{(\sqrt{b_m} - \sqrt{b_k})^2}{1 + b_m} g_k,$$

where

$$C_k = 8Mg_k \frac{1 + b_k}{\Delta_k^2(b)}.$$

Using this inequality, we obtain that

$$I'_n \leq C_k \sum_{m=0}^{\infty} \frac{(\sqrt{b_m} - \sqrt{b_k})^2}{1 + b_m} g_m T_{k,m}^{(n)}. \quad (22)$$

By condition (20), the right-hand side of (22) tends to zero as $n \rightarrow \infty$ for any fixed k . By the conditions on b_k and g_k , this part has the majorant $\alpha_{k,n}$, such that $\limsup_{k \rightarrow \infty} \alpha_{k,n}^{\frac{1}{k}} = 1$ for any fixed n .

Consider now I''_n . Obviously,

$$I''_n \leq Mg_k \left\{ \sum_{m \neq k} T_{k,m}^{(n)} + |T_{k,k}^{(n)} - 1| \right\}.$$

By (21), we have the following inequality:

$$\sum_{m \neq k} T_{k,m}^{(n)} \leq 4 \frac{1 + b_k}{\Delta_k^2(b)} \sum_{m \neq k} T_{k,m}^{(n)} \frac{(\sqrt{b_m} - \sqrt{b_k})^2}{1 + b_m} g_m.$$

The right-hand side of the last inequality coincides with the right-hand side of (22). It remains to estimate only the term $g_k |T_{k,k}^{(n)} - 1|$. From condition (20) and Theorem B, it follows that

$$\left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} \frac{g_m}{1 + b_m} - \frac{g_k}{1 + b_k} \right| \leq t_{n,k},$$

where $t_{n,k}$ tends to zero as $n \rightarrow \infty$ for any fixed k and $\limsup_{k \rightarrow \infty} t_{n,k}^{\frac{1}{k}} = 1$ for each n . Therefore

$$g_k |T_{k,k}^{(n)} - 1| \leq (1 + b_k) \left\{ t_{n,k} + \frac{1}{\Delta_k^2} \sum_{m \neq k} T_{k,m}^{(n)} \frac{(\sqrt{b_m} - \sqrt{b_k})^2}{1 + b_m} g_m \right\}.$$

Using all these estimates in (9), we complete the proof. \square

Let us note that this theorem has many corollaries. In particular, choosing $g_k = 1 + b_k$, we obtain a theorem on convergence in the subspace of $A(D)$ of functions with coefficients $|f_k| \leq M(1 + b_k)$, and in the case $b_k = k^2$, $g_k = 1$ in the space of functions with bounded coefficients.

Corollary 1. Formula (19) is valid for any function $f \in A_g(D)$ with $g_k = 1 + k^2$ if and only if

$$\lim_{n \rightarrow \infty} \left\| T_n \frac{1}{(1 - \phi(z))^v} - \frac{1}{(1 - \phi(z))^v} \right\|_{A(D),r} = 0, \quad v = 1, 2, 3.$$

Corollary 2. Let $c_k = \frac{g_k}{1+k^2}$, $D = \{|z| < 1\}$ and

$$C(z) = \sum_{k=0}^{\infty} c_k z^k.$$

Then (19) is valid for each function $f \in A_g(D)$ if and only if

$$\lim_{n \rightarrow \infty} \|T_n z^v C^{(v)}(z) - z^v C^{(v)}(z)\|_{A(D),r} = 0, \quad v = 0, 1, 2.$$

Theorem 5. Let

$$a(z) = \sum_{k=0}^{\infty} a_k \phi^k(z).$$

If the sequence of linear k -positive operators $T_n : A(D) \rightarrow A(D)$ satisfies the condition

$$\lim_{n \rightarrow \infty} \|T_n \phi^v(z) a^{(v)}(z) - \phi^v(z) a^{(v)}(z)\|_{A(D),r} = 0, \quad v = 0, 1, 2, \quad (23)$$

then (19) holds for each function $f \in A_g(D)$, with $g_k = a_k$, $k = 0, 1, 2, \dots$

Proof. From condition (23) and Theorem A it follows that

$$\sum_{m=0}^{\infty} (m-k)^2 a_m T_{k,m}^{(n)} \leq \varepsilon_n (1 + \delta_n)^k (1+k)^2, \quad (24)$$

where ε_n and δ_n tend to zero as $n \rightarrow \infty$. If $f \in A_g(D)$ with $g_k = a_k$, then

$$\begin{aligned} \|T_n f - f\|_{A(D),r} &\leq 2M \left\{ \sum_{k=0}^{\infty} r^k a_k \sum_{m=0}^{\infty} (m-k)^2 a_m T_{k,m}^{(n)} + \sum_{k=0}^{\infty} r^k a_k \left| \sum_{m=0}^{\infty} T_{k,m}^{(n)} - 1 \right| \right\} \\ &= 2M \{S'_n + S''_n\}. \end{aligned}$$

By (24), we have

$$S'_n \leq \varepsilon_n \sum_{k=0}^{\infty} r^k a_k (1+k)^2 (1+\delta_n)^k.$$

Thus $S'_n \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, (24) gives

$$\sum_{m \neq k} T_{k,m}^{(n)} \leq \varepsilon_n (1 + \delta_n)^k (1+k)^2. \quad (25)$$

Using condition (23) with $v = 0$, we obtain that

$$a_k |T_{k,k}^{(n)} - 1| \leq \varepsilon_n (1 + \delta_n)^k + \sum_{m \neq k} a_k (m-k)^2 T_{k,m}^{(n)}.$$

Due to (24), we have

$$a_k |T_{k,k}^{(n)} - 1| \leq 2\varepsilon_n (1 + \delta_n)^k (1+k)^2. \quad (26)$$

Therefore, (25) and (26) imply that

$$S_n'' \leq \varepsilon_n \sum_{k=0}^{\infty} r^k a_k (1+k)^2 (1+\delta_n)^k + 2\varepsilon_n \sum_{k=0}^{\infty} r^k (1+k)^2 (1+\delta_n)^k$$

and $S_n'' \rightarrow 0$ as $n \rightarrow \infty$. \square

In conclusion, we give an application of Theorem 2 to a particular sequence of linear k -positive operators.

Suppose that $f \in A_g(D)$ and $g_k = 1+k^2$. Consider the sequence of linear k -positive operators

$$T_n f(z) = f(z) + \frac{1}{n} \sum_{k=0}^n \phi^k(z) \sum_{m=0}^n \binom{n}{m} \left(\frac{k}{n}\right)^m \left(1 - \frac{k}{n}\right)^{n-m} f_m$$

such that, for these operators,

$$T_{k,m}^{(n)} = \delta_{k,m} + \begin{cases} \frac{1}{n} \binom{n}{m} \left(\frac{k}{n}\right)^m \left(1 - \frac{k}{n}\right)^{n-m}, & \text{if } k \leq n, m \leq n, \\ 0, & \text{for other } k, m. \end{cases}$$

Following Theorem 2, we calculate (15) for $\tilde{g}_v(z)$ defined as in (13) with $h_k = k^2$. We have

$$\begin{aligned} T_n \tilde{g}_0(z) &= \tilde{g}_0(z) + \frac{1}{n} \sum_{k=0}^n \phi^k(z) \sum_{m=0}^n \binom{n}{m} \left(\frac{k}{n}\right)^m \left(1 - \frac{k}{n}\right)^{n-m} \\ &= \tilde{g}_0(z) + \frac{1}{n} \sum_{k=0}^n \phi^k(z), \\ T_n \tilde{g}_1(z) &= \tilde{g}_1(z) + \frac{1}{n} \sum_{k=0}^n \phi^k(z) \sum_{m=0}^n \binom{n}{m} \left(\frac{k}{n}\right)^m \left(1 - \frac{k}{n}\right)^{n-m} m \\ &= \tilde{g}_1(z) + \frac{1}{n} \sum_{k=0}^n k \phi^k(z), \\ T_n \tilde{g}_2(z) &= \tilde{g}_2(z) + \frac{1}{n} \sum_{k=0}^n \phi^k(z) \sum_{m=0}^n \binom{n}{m} \left(\frac{k}{n}\right)^m \left(1 - \frac{k}{n}\right)^{n-m} m^2 \\ &= \tilde{g}_2(z) + \frac{1}{n} \frac{n(n-1)}{n^2} \sum_{k=0}^n k^2 \phi^k(z) + \frac{1}{n} \sum_{k=0}^n k \phi^k(z). \end{aligned}$$

Since

$$\begin{aligned} \sum_{k=0}^{\infty} \phi^k(z) &= \frac{1}{1-\phi(z)}, \quad \sum_{k=0}^{\infty} k \phi^k(z) = \frac{\phi(z)}{(1-\phi(z))^2}, \\ \sum_{k=0}^{\infty} k^2 \phi^k(z) &= \frac{\phi^2(z)(1+\phi(z))}{(1-\phi(z))^3}, \end{aligned}$$

we have

$$\lim_{n \rightarrow \infty} \|T_n \tilde{g}_v - \tilde{g}_v\|_{A(D),r} = 0, \quad v = 0, 1, 2,$$

and hence $T_n f$ converges to f in $A(D)$ for any function $f \in A_g(D)$ with $g_k = 1+k^2$.

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References

- [1] R.A. Ahadov, On the convergence of sequences of linear operators in a space of functions that are analytic in the disc, *Izv. Akad. Nauk Azerb. SSR Ser. Fiz.- Tekh. Mat. Nauk* 2 (1) (1981) 67–71 (in Russian).
- [2] F. Altomare, M. Campiti, Korovkin-Type Approximation Theory and its Applications, in: *de Gruyter Studies in Mathematics*, vol. 17, Walter de Gruyter and Co., Berlin, 1994.
- [3] O. Duman, Statistical approximation theorems by k -positive linear operators, *Arch. Math. (Basel)* 86 (6) (2006) 569–576.
- [4] M.A. Evgrafov, The method of near systems in the space of analytic functions and its application to interpolation, *Tr. Mosk. Mat. Obs.* 5 (1956) 89–201 (in Russian); *Amer. Math. Soc. transl.* 16 (1960) 195–314 (English transl.).
- [5] A.D. Gadjiev, Linear k -positive operators in a space of regular functions, and theorems of P.P. Korovkin type, *Izv. Akad. Nauk Azerb. SSR Ser. Fiz.- Tekh. Mat. Nauk* 5 (1974) 49–53 (in Russian).
- [6] A.D. Gadjiev, Theorems of Korovkin type, *Mat. Zametki* 20 (5) (1976) 781–786; *Math. Notes* 20 (1976) 995–998 (English transl.).
- [7] A.D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* 32 (1) (2002) 129–138.
- [8] A.D. Gadjiev, The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous of that P.P. Korovkin, *Soviet Dokl. Akad. Nauk SSSR* 218 (1974) 1001–1004; *Soviet Math. Dokl.* 15 (1974) 1433–1436 (English transl.).
- [9] H.H. Hacısalıhoğlu, A.C. Hacıyev, On convergence of sequences of linear positive operators, Ankara, 1995 (in Turkish).
- [10] N. İspir, Convergence of sequences of k -positive linear operators in subspaces of the space of analytic functions, *Hacet. Bull. Nat. Sci. Eng. Ser. B* 28 (1999) 47–53.
- [11] N. İspir, Ç. Atakut, On the convergence of a sequence of positive linear operators on the space of m -multiple complex sequences, *Hacet. Bull. Nat. Sci. Eng. Ser. B* 29 (2000) 47–54.
- [12] M. Khaplanov, Linear operators in analytic space, *Rostov. Gos. Univ. Uch. Zap. Fiz.-Mat. Fak.* 43 (1959) 83–118 (in Russian).